Distribution of time-headways in a particle-hopping model of vehicular traffic

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Some prototype models of interacting particles driven far from equilibrium are known to capture many qualitative features of vehicular traffic. The *time headway* is defined as the time interval between two successive vehicles recorded by a detector placed at a fixed position on the highway. We report analytical calculation of the distributions of the time headways in some special cases of the Nagel-Schreckenberg model, which is a particle-hopping model of vehicular traffic on idealized single-lane highways. We also present numerical results for the time-headway distribution in more general situations in this model. [S1063-651X(98)11808-1]

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I. INTRODUCTION

The techniques of fluid dynamics and statistical mechanics have been powerful tools in theoretical investigations of a wide variety of problems in the science and engineering of vehicular traffic [1–9]. The main aim of this Brief Report is to calculate the distribution of the *time headways* (TH's) in a particle-hopping model, namely, the *Nagel-Schreckenberg* (NS) model [4] of vehicular traffic on idealized single-lane highways, where the TH is defined as the time interval between the departures (or arrivals) of two successive vehicles recorded by a detector placed at a fixed position on the highway [1]. This distribution is not merely of academic interest to statistical physicists, but is also of practical interest to traffic engineers [1].

II. MODEL

In the particle-hopping models a lane is represented by a one-dimensional lattice of L sites. Each of the lattice sites can be either empty or occupied by at most one ''vehicle.'' If periodic boundary condition is imposed, the density c of the vehicles is N/L, where $N(\leq L)$ is the total number of vehicles. In the NS model [4] the speed V of each vehicle can take one of the $V_{\max}+1$ allowed *integer* values $V=0,1,\ldots,V_{\max}$. Suppose V_n is the speed of the nth vehicle at time t. At each *discrete time* step $t\rightarrow t+1$, the arrangement of N vehicles is updated *in parallel* according to the following ''rules.''

Step 1: Acceleration. If, $V_n<V_{\max}$, the speed of the nth vehicle is increased by 1, i.e., $V_n\rightarrow V_n+1$.

Step 2: Deceleration (due to other vehicles). If d_n is the gap in between the nth vehicle and the vehicle in front of it, and if $d_n\leq V_n$, the speed of the n-th vehicle is reduced to d_n-1, i.e., $V_n\rightarrow d_n-1$.

Step 3: Randomization. If $V_n>0$, the speed of the nth vehicle is decreased randomly by unity (i.e., $V_n\rightarrow V_n-1$) with probability p $(0\leq p\leq 1)$; p, the random deceleration probability, is identical for all the vehicles and does not change during the updating.

Step 4: Vehicle movement. Each vehicle is moved forward so that $X_n\rightarrow X_n+V_n$ where X_n denotes the position of the nth vehicle at time t.

The nonvanishing braking probability p is essential for a realistic modeling of traffic flow [5] and, therefore, the NS model may be regarded as stochastic cellular automata [6]. An effectively free flow of traffic takes place when the density of vehicles is sufficiently low, whereas high density leads to congestion and traffic jams. The density c_m corresponding to the maximum flux is usually called the *optimum* density.

In the NS model, the number of empty lattice sites in front of a vehicle is taken to be a measure of the corresponding distance headway; the exact analytical expression for the distribution of these distance headways in the steady state has been calculated for $V_{\max}=1$ [8,9]. There are several earlier papers, published by statisticians and traffic engineers, where the form of the TH distribution was derived on the basis of heuristic arguments [10]. However, to our knowledge, no derivation of the TH distribution in the particle-hopping models has been reported so far in the literature.

III. RESULTS AND DISCUSSION

A. Analytical calculations for $V_{\max}=1$

For the convenience of our analytical calculations, we change the order of the steps in the update rules in a manner that does not influence the steady-state properties of the model. Following Schreckenberg *et al.* [7], we assume a sequence of steps 2-3-4-1, instead of 1-2-3-4; the advantage is that there is no vehicle with $V=0$ immediately after the acceleration step. Consequently, if $V_{\max}=1$, we can then use a binary site variable σ to describe the state of each site; $\sigma=0$ represents an empty site and $\sigma=1$ represents a site occupied by a vehicle whose speed is unity.

We label the position of the detector by $j=0$, the site immediately in front of it by $j=1$, and so on. The detector clock resets to $t=0$ everytime a vehicle leaves the detector site. We begin our analytical calculations for $V_{\max}=1$ by writing $\mathcal{P}(t)$, the probability of a TH t between a ''leading'' vehicle (LV) and the ''following'' vehicle (FV) as

$$\mathcal{P}(t)=\sum_{t_1=1}^{t-1} P(t_1)Q'(t-t_1|t_1), \tag{1}$$

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where $P(t_1)$ is the probability that there is a time interval t_1 between the departure of the LV and the arrival of the FV at the detector site, and $Q'(t-t_1|t_1)$ is the conditional probability that the FV halts for $t-t_1$ time steps when it arrived at the detector site t_1 time steps after the departure of the LV.

1. Analytical calculations for $V_{max}=1$: The factorization approximation

Strictly speaking, there are correlations between the time of arrival t_1 and the halting time $t-t_1$. However, for the time being, we make the ''factorization approximation'' under which the expression (1) for $\mathcal{P}(t)$ can be written as

$$\mathcal{P}(t)=\sum_{t_1=1}^{t-1} P(t_1)Q(t-t_1), \tag{2}$$

where $Q(t-t_1)$ is the probability that the FV halts at the detector site for $t-t_1$ time steps; the leading corrections to this approximation, arising from the correlations, will be incorporated later in Sec. III A 2.

To calculate $P(t_1)$, we have to consider spatial configurations at $t=0$ for which the FV can reach the detector site within t_1 steps. The configurations of interest are thus of the form $(\underbrace{100\cdots 0}_{n}|\underline{0})$ where $n=1,2,\ldots t_1$. The underlined zero implies that we have to find the conditional steady-state probability for the given configuration subject to the condition that the underlined site (detector site) is empty. This probability, $\Pi(n)$, in the two-cluster approximation is given by [7,9]

$$\Pi(n)=\mathcal{C}(1|\underline{0})\{\mathcal{C}(0|\underline{0})\}^{n-1}, \tag{3}$$

where $\mathcal{C}$ gives the two-cluster steady-state configurational probability for the argument configuration, and the underlined symbols imply the conditional, as usual. The expressions for the various $\mathcal{C}$'s are given by [7,9]

$$\mathcal{C}(0|\underline{0})=\mathcal{C}(\underline{0}|0)=1-\frac{y}{d}, \tag{4}$$

$$\mathcal{C}(0|\underline{1})=\mathcal{C}(\underline{1}|0)=\frac{y}{c}, \tag{5}$$

$$\mathcal{C}(1|\underline{0})=\mathcal{C}(\underline{0}|1)=\frac{y}{d}, \tag{6}$$

$$\mathcal{C}(1|\underline{1})=\mathcal{C}(\underline{1}|1)=1-\frac{y}{c}, \tag{7}$$

where

$$y=\frac{1}{2q}(1-\sqrt{1-4qcd}), \tag{8}$$

$q=1-p$ and $d=1-c$. For all configurations with $t_1>n$, t_1-1 time steps elapse in crossing $n-1$ bonds (as the last bond is crossed certainly at the last time step). Thus

$$P(t_1)=\sum_{n=1}^{t_1} \Pi(n)q^n p^{t_1-n}\,{}^{t_1-1}C_{n-1}. \tag{9}$$

Using expression (3) for $\Pi(n)$ in Eq. (9) we obtain

$$P(t_1)=\mathcal{C}(1|\underline{0})q[\mathcal{C}(0|\underline{0})q+p]^{t_1-1}. \tag{10}$$

Next we calculate $Q(t_2)$ by considering all configurations of the form $(\underline{1}|\underbrace{11\cdots 1}_{m-1}0)$ at $t=t_1$. Here, the underlined part implies that the detector site is occupied by the FV, and we consider configurations with a queue of $(m-1)$ vehicles ahead of it and the foremost vehicle of this queue has an empty site ahead. The steady-state probability of this configuration is given by

$$\Pi'(m)=\mathcal{C}(\underline{1}|\underbrace{11\cdots 1}_{m-1}0)=\{\mathcal{C}(\underline{1}|1)\}^{m-1}\mathcal{C}(\underline{1}|0) \tag{11}$$

in the two-cluster approximation. To find all possible configurations of this type that contribute to the waiting time of t_2 time steps, we have to consider queue sizes up to t_2 $(m\leqslant t_2)$. Thus, using expression (11) for $\Pi'(m)$, we obtain

$$\begin{aligned}Q(t_2)&=\sum_{m=1}^{t_2} \Pi'(m)q^m p^{t_2-m}\,{}^{t_2-1}C_{m-1}\\ &=\mathcal{C}(\underline{1}|0)q[q\mathcal{C}(\underline{1}|1)+p]^{t_2-1}.\end{aligned} \tag{12}$$

The TH distribution, $\mathcal{P}(t)$, can now be calculated using Eqs. (2), (10), and (12), and is given by

$$\begin{aligned}\mathcal{P}(t)&=q^2\mathcal{C}(1|\underline{0})\mathcal{C}(\underline{1}|0)\left[\frac{\{\mathcal{C}(0|\underline{0})q+p\}^{t-1}-\{q\mathcal{C}(\underline{1}|1)+p\}^{t-1}}{\{\mathcal{C}(0|\underline{0})q+p\}-\{q\mathcal{C}(\underline{1}|1)+p\}}\right]\\ &=\left(\frac{yq}{c-d}\right)\left[\left\{1-\left(\frac{yq}{c}\right)\right\}^{t-1}-\left\{1-\left(\frac{yq}{d}\right)\right\}^{t-1}\right],\end{aligned} \tag{13}$$

where the final result [Eq. (13)] was obtained using Eqs. (4)–(7) and y is given by Eq. (8).

2. Analytical calculations for $V_{max}=1$: Beyond factorization approximation

In approximating $Q'(t-t_1|t_1)$ by $Q(t-t_1)$, we ignored the fact that, for $V_{\max}=1$, the LV is *certainly* present at $j=1$ at $t=0$. If the LV is still at site $j=1$ at $t>t_1$, when the FV is at $j=0$, it hinders the forward movement of the FV. Therefore, the exact TH distribution is given by Eq. (1), where $P(t_1)$ is given by Eq. (10) and the conditional probability $Q'(t-t_1|t_1)$ is given by

$$Q'(t-t_1|t_1)=\sum_{t_1=1}^{t-1}\{qp^{t-t_1-1}S(t_1)+qR(t_1,t-t_1)\} \tag{14}$$

where

$$S(t_1)=Q(1)+Q(2)+\cdots+Q(t_1)=\sum_{t_3=1}^{t_1} Q(t_3) \quad (15)$$

and

$$R(t_1,t_2)=Q(t_1+1)p^{t_2-2}+Q(t_1+2)p^{t_2-3}+\cdots+Q(t_1+t_2-1)=\sum_{t_3=t_1+1}^{t_1+t_2-1} Q(t_3)p^{t_1+t_2-t_3-1}, \quad (16)$$

$Q(t_2)$ being given by Eq. (12). The first series on the right-hand side of Eq. (14) accounts for the situation where the LV hops out of the site $j=1$ before the FV arrives at the site $j=0$, and the second series accounts for the situation where the FV arrives at $j=0$ before the LV hops out of the site $j=1$. The general term in Eq. (16) means that the LV halts for t_3 time steps ($t_3>t_1$), the FV being blocked at site $j=0$ for (t_3-t_1) time steps, contributing only a factor of $Q(t_3)$. After the LV hops out of site $j=1$ (at $t=t_3$), the FV continues to halt at site $j=0$ upto $t=t_1+t_2$ by braking, picking up a factor $p^{t_1+t_2-t_3-1}$.

Using expression (12) for $Q(t_3)$ in Eqs. (15) and (16) we obtain

$$S(t_1)=1-[\mathcal{C}(\underline{1}|1)q+p]^{t_1} \quad (17)$$

and

$$R(t_1,t_2)=\frac{\mathcal{C}(\underline{1}|0)}{\mathcal{C}(\underline{1}|1)}[\mathcal{C}(\underline{1}|1)q+p]^{t_1}\times\{[\mathcal{C}(\underline{1}|1)q+p]^{t_2-1}-p^{t_2-1}\}. \quad (18)$$

So, finally, using Eqs. (17), (18) and (14) in Eq. (1), we obtain the total probability distribution

$$\mathcal{P}(t)=q\left[\frac{\mathcal{C}(\underline{1}|0)}{\mathcal{C}(\underline{1}|1)}\right]A^{t-1}+q\left[\frac{\mathcal{C}(1|\underline{0})}{\mathcal{C}(0|\underline{0})}\right]B^{t-1}-q\left[\frac{\mathcal{C}(\underline{1}|0)}{\mathcal{C}(\underline{1}|1)}+\frac{\mathcal{C}(1|\underline{0})}{\mathcal{C}(0|\underline{0})}\right]p^{t-1}-q^2\left[\frac{\mathcal{C}(1|\underline{0})}{\mathcal{C}(\underline{1}|1)}\right](t-1)Ap^{t-2} \quad (19)$$

where $A=1-q\mathcal{C}(\underline{1}|0)$ and $B=1-q\mathcal{C}(1|\underline{0})$. In terms of p,q,c,d, and y the TH distribution $\mathcal{P}(t)$ can be written as

$$\mathcal{P}(t)=\left[\frac{qy}{c-y}\right]\{1-(qy/c)\}^{t-1}+\left[\frac{qy}{d-y}\right]\{1-(qy/d)\}^{t-1}-\left[\frac{qy}{c-y}+\frac{qy}{d-y}\right]p^{t-1}-q^2(t-1)p^{t-2}. \quad (20)$$

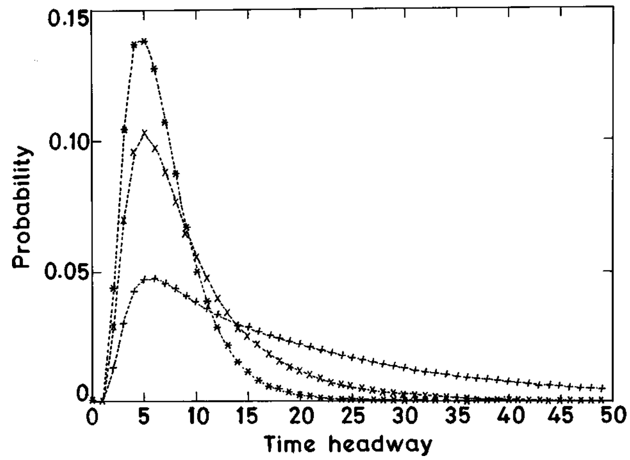

FIG. 1. The time-headway distribution in the NS model with $V_{max}=1$ ($p=0.5$) for the densities $c=0.1(+)$, $c=0.25(\times)$, and $c=0.5(*)$. The continuous curves represent the analytical result, while the discrete data points have been obtained from computer simulation.

Note that, in contrast to the approximate expression (13), the exact expression (20) of $\mathcal{P}(t)$ possesses the well-known particle-hole symmetry [4] in the NS model for $V_{max}=1$, i.e., the expression for the densities c and $1-c$ are identical (see Fig. 1).

B. Numerical results for $V_{max}>1$

As analytical calculations are too complicated to carry through for $V_{max}>1$, we have computed the TH distributions for all $V_{max}>1$ only through computer simulation. In Fig. 2 we present our numerical data for the TH distribution in the NS model with $V_{max}=5$, as several earlier works have demonstrated that this particular choice of V_{max} leads to quite realistic qualitative descriptions of some other features of vehicular traffic on highways. The qualitative features of the distribution in Fig. 2 are similar to those in Fig. 1, except the breakdown of particle-hole symmetry for all $V_{max}>1$. In addition, $\mathcal{P}(t=0)=0$, but $\mathcal{P}(t=1)$ need not vanish when

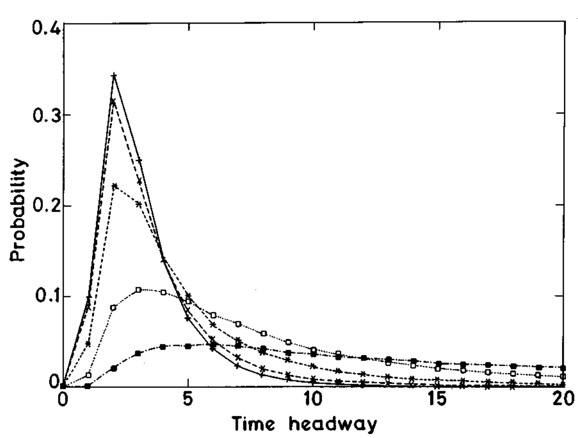

FIG. 2. The time-headway distribution in the NS model with $V_{max}=5$ ($p=0.5$) for the densities $c=0.10(+)$, $0.25(\times)$, $0.50(*)$, $0.75(\square)$, and $0.90(\blacksquare)$. The discrete data points have been obtained from computer simulation, while the continuous curves are merely guides to the eye.

$V_{\max}=5$ in contrast to the fact that $\mathcal{P}(t\leqslant 1)=0$, irrespective of the density of the vehicles, when $V_{\max}=1$.

Recall that the flux q of the vehicles can be written as $q=N/T$ where $T=\Sigma_{i=1}^{N}t_i$ is the sum of the TH recorded for all the N vehicles. One can rewrite q as $q=1/[(1/N)\Sigma_i t_i]=1/T_{\text{av}}$ where T_{av} is the average TH. Therefore, T_{av} is expected to exhibit a minimum, just as q is known to exhibit a maximum, at $c=c_m$ with the variation of density c of the vehicles; this is consistent with one's intuitive expectation that both at very low and very high densities there are long time gaps in between the departures of two successive vehicles from a given site.

We have plotted the most probable TH, T_{mp}, as a function of the density c in Fig. 3; the trend of variation of T_{mp} with c is very similar to that of T_{av}. The particle-hole symmetry of the T_{mp} versus c curve also breaks down for all $V_{\max}>1$.

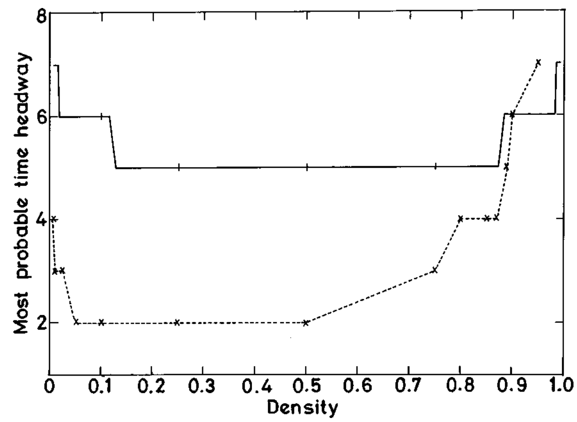

FIG. 3. The most probable time-headway plotted against the density of the vehicles in the NS model. The full line, obtained from expression (20), corresponds to $V_{\max}=1$, and the symbol + represents the corresponding numerical data obtained from our computer simulation. The discrete data points, represented by the symbol $\times$, correspond to $V_{\max}=5$, and have been obtained from computer simulation; the dotted line joining these data points serves merely as a guide to the eye.

IV. SUMMARY AND CONCLUSION

In this Brief Report we have derived the TH distribution only for the original version of the NS model of vehicular traffic on single-lane highways. Equation (20), which is the main analytical result of this Brief Report, is the exact expression for the TH distribution in this model when $V_{\max}=1$. However, for $V_{\max}>1$, the TH distributions in this model have been obtained by carrying out computer simulation and are, therefore, approximate. Nevertheless, our results demonstrate that, in spite of being only a minimal model, the NS model captures the essential qualitative features of the TH distribution of vehicular traffic on highways [1]. But, in order to make a direct *quantitative* comparison with empirical data from the highway traffic the TH distribution will have to be computed by incorporating some of the recent generalizations and extensions [3] of the NS model proposed recently in the literature.

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